second sphere,

$$MQ = 2bl - 2kn.$$

If R be the mid-point of PQ, then

$$MR = \frac{MP + MQ}{2} = (a + b)l - 2kn.$$

Hence, if in (3) we set

$$r = (a + b)l - 2kn, \tag{4}$$

we readily obtain expressions for the coördinates x, y, z of the middle point R of the segment determined by the two spheres on the line through $(0, 0, k)$ and having the direction (l, m, n). To find the equation of the locus of R for all directions of the line we must eliminate l, m, n, which we do as follows:

From (3), we have,

$$x^2 + y^2 + (z - k)^2 = r^2; \tag{5}$$

and from (4) and (3),

$$r = (a + b)\left(\frac{x}{r}\right) - 2k\left(\frac{z - k}{r}\right),$$

or

$$r^2 = (a + b)x - 2k(z - k). \tag{6}$$

Equating the values of r^2 given by (5) and (6), we have for the equation of the locus of R,

$$x^2 + y^2 + z^2 - (a + b)x = k^2. \tag{7}$$

The locus is, therefore, a sphere whose center is midway between the centers of the given spheres and which contains all points common to these spheres. It is evident then that the choice of any point other than M common to the two spheres would lead to the same locus.

Also solved by the PROPOSER.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, Washington, D. C.

Determine a function of x independent of b, such that

$$\int_b^{b+1} f(x)dx = \frac{1}{b + 1},$$

the real part of b being positive.

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

There is one solution only satisfying the conditions:
(i) $f(x)$ is one-valued and continuous in the right half plane of complex numbers, and
(ii) $f(x + n)$ approaches 0 as n approaches $+\infty$ through integral values for every x.
Let $f(x)$ be such a solution. Differentiating the equation

$$\int_b^{b+1} f(x)dx = \frac{1}{b + 1}, \tag{1}$$

we get, after substituting x for b,

$$f(x + 1) - f(x) = -\frac{1}{(x + 1)^2}.$$

Similarly,

$$\begin{aligned} f(x + 2) - f(x + 1) &= -\frac{1}{(x + 2)^2}, \\ &\vdots \\ f(x + n) - f(x + n - 1) &= -\frac{1}{(x + n)^2}. \end{aligned}$$

Adding these equations, we have

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \cdots + \frac{1}{(x+n)^2} + f(x+n).$$

Letting n approach $+\infty$, we get, on account of (ii),

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \cdots \text{ to infinity.} \tag{2}$$

To show that (2) is a solution of (1), we observe that the series converges uniformly with respect to x over the right half plane including the axis of imaginaries, since $|x+n| \geqq n$.

Hence,

$$\int_b^{b+1} f(x)dx = \left(\frac{1}{b+1} - \frac{1}{b+2}\right) + \left(\frac{1}{b+2} - \frac{1}{b+3}\right) + \cdots = \frac{1}{b+1},$$

for any path of integration lying in the above region.

The series (2) may be summed by means of the gamma function. Thus integrating between the limits from 0 to x, we have

$$\int_0^x f(x)dx = \left(\frac{1}{1} - \frac{1}{x+1}\right) + \left(\frac{1}{2} - \frac{1}{x+2}\right) + \cdots = S, \text{ say.}$$

Since S is uniformly convergent,

$$\int_0^x Sdx = \left\{\frac{x}{1} - \log\left(1 + \frac{x}{1}\right)\right\} + \left\{\frac{x}{2} - \log\left(1 + \frac{x}{2}\right)\right\} + \cdots$$

$$= -\log\left[\left\{e^{-(x/1)}\left(1 + \frac{x}{1}\right)\right\}\left\{e^{-(x/2)}\left(1 + \frac{x}{2}\right)\right\}\cdots\right]$$

$$= -\log\left(\frac{1}{\Gamma(x)xe^{\gamma x}}\right) = \log\Gamma(x+1) + \gamma x.$$

Hence,

$$f(x) = \frac{dS}{dx} = \frac{d^2}{dx^2}[\log\Gamma(x+1)].$$

The general solution of (1) subject to (i) but not to (ii) is now seen to be

$$f(x) = \frac{d^2}{dx^2}\log\Gamma(x+1) + \frac{d}{dx}\phi(x),$$

where $\phi(b+1) = \phi(b)$, and, therefore, $\phi(x)$ is any analytic function admitting the period 1.

Also solved by W. R. Ransom and the Proposer.

414. Proposed by C. N. SCHMALL, New York City.

Among spherical triangles having the same base and equal altitudes, show that the isosceles triangle has the greatest vertical angle. Show that this is also true for plane triangles.

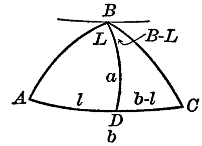
Solution by W. J. Thome, University of Detroit.

ABC is any spherical triangle. BD is a great circle arc perpendicular to AC. The angle B is broken up into two parts, L and $(B - L)$, and the corresponding opposite sides are l and $(b - l)$. Suppose B moves around on a small circle concentric with AC. Then BD remains constant in value.

In the $\triangle ABD$, $\cot L = \sin a \cot l$, and in the $\triangle BDC$, $\cot(B - L) = \sin a \cot(b - l)$. Now $B = L + (B - L) = \cot^{-1}(\sin a \cot l) + \cot^{-1}(\sin a \cot(b - l))$.

Differentiating,

$$\frac{dB}{dl} = \sin a\left[\frac{-\csc^2 l}{1 + [\sin a \cot l]^2} + \frac{\csc^2(b-l)}{1 + [\sin a \cot(b-l)]^2}\right].$$



which is 0 if $b = 2l$. This evidently gives a maximum value of B. But if $b = 2l$, $AB = BC$ and the $\triangle$ is isosceles.